Virtual electron diffusion through a quantum dot in the presence of electromagnetic fluctuations

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We consider a double junction system in the Coulomb blockade regime with an island with discrete energy levels and analyze the effect of electromagnetic fluctuations on elastic cotunneling. We obtain the analytic expression for the elastic cotunneling current at zero temperature, which shows the power-law suppression of cotunneling at low voltages due to circuit impedance $I \sim V^{1+2\,\mathrm{Re}\,Z(0)/R_K}$. The results will be useful for calculating the accuracy of single-electron devices used in metrology.

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I. INTRODUCTION

A proposal for a new International System of Units (SI) is currently being considered by the metrology community.[1] According to this proposal, the new SI (sometimes referred to as quantum SI) would be based on a set of exactly defined values of fundamental constants. Before this system is adopted, an important issue regarding the value of Planck's constant, h, needs to be resolved. Namely, the value of h obtained from the watt balance experiment is significantly lower than the value deduced from combined x-ray and optical interferometry measurements of the lattice spacing of a single silicon crystal d_{220}, together with the measured value of the molar volume of silicon $V_m(\mathrm{Si})$ (fractional difference of $\sim 10^{-6}$).[2,3] The virtual electron transfer through a small metallic island that we consider in this paper is important for the level of confidence that we can have in the new SI. We review below some important aspects of this proposal in order to put the problem in the context.

One of the underlying assumptions of the proposed quantum SI, which is used in fixing the value of h, is that the expression for the Josephson constant,

$$K_J = 2e/h, \tag{1}$$

is the exact relation in terms of e and h. Since the discovery of the Josephson effect,[4] a considerable amount of work has supported such an assumption. By considering a superconducting ring, Bloch[5] arrived at the Josephson result [Eq. (1)] without relying on the Bardeen-Cooper-Schrieffer (BCS) or Ginzburg-Landau (GL) theories but by using general arguments such as gauge transformation and time-reversal invariance, and the requirement that the wave function be single valued along the ring. Similarly, by considering an infinitely long, one-dimensional Josephson junction, Fulton[6] showed that any temperature, material, frequency, or voltage dependence of K_J would violate Faraday's law. The universality of K_J has been tested experimentally, for example by Tsai *et al.*[7] for different types of conventional metallic superconductors (with uncertainty of the order 10^{-16}) and by Klushin *et al.*[8] by comparing arrays of metallic superconductors and oxide high-temperature superconductor (HTS) junctions (with uncertainty of the order 10^{-8}).

Another assumption included in the proposed new SI is that the von Klitzing constant, given by

$$R_K = h/e^2, \tag{2}$$

is also an exact relation. After the discovery of the quantum Hall effect,[9] Laughlin[10] put forward an argument to explain Eq. (2) by interpreting idealized looped ribbon of two-dimensional electron gas with a perpendicular magnetic field as a quantum pump between reservoirs connected to the edges of the ribbon. Various authors have generalized his argument by, for example, including the topological quantum numbers,[11] or more realistic boundary conditions.[12] An experimentally obtained value of R_K, for extrapolated longitudinal resistivity $\rho_{xx}=0$, has been found to be independent of material and other device specifications with an uncertainty level of about 10^{-10}.[13,14]

Direct measurements in terms of the present SI units of V and Ω of both K_J and R_K have also been performed using calculable capacitor. Using the results for e and h from other areas of physics, expressions (1) and (2) are supported, with an uncertainty of about 10^{-7}.[2]

The third effect that also plays role in the proposed new SI is the single-electron tunneling (SET) effect (for an early review of this field see, for example, the paper by Averin and Likharev[15]). One possibility is to use SET devices for the realization of the ampere at low currents through the basic equation $I=ef$ from the defined value of electron charge e at the applied gate frequency f determined from atomic clocks. Although promising results for metrology have been achieved, for example for the cryogenic capacitor standard,[16] SET devices have not yet been used to the same extent in the present SI as devices using the other two effects above. They have also not been used in adjustments of fundamental constants. Since the SET devices are elaborate structures from elementary particles perspective, the question that arises[17] is whether there should be many-body or quantum electrodynamics (QED) corrections, that is, whether elementary charge is in fact being transferred through the circuit. This question could be asked for both other effects as well. An argument against the corrections has been given by Langenberg and Schrieffer.[18] Also, the value for the inverse fine structure constant inferred from the quantum Hall measurements, $\alpha^{-1}=2R_K/\mu_0 c$, is within the experimental uncertainty at level of 10^{-8}, in agreement[3] with the QED calculations of the anomalous magnetic moment of the electron.[19] Nevertheless, this question is still relevant when, for example, there are competing processes in device operation, with different tunneling times, such as sequential SET and elastic cotunneling.

Due to the importance of the above effects in the new SI, in particular for confidence in the exact value being assigned to h, it is of interest to perform additional checks of the assumptions of their exactness. One such test, the so-called quantum metrological triangle, first proposed by Likharev and Zorin,[20] is being pursued by several groups (for current status see a recent review by Keller[17]). It provides a direct test of the consistency of the fundamental constants involved in the above effects. The device that is investigated for this purpose by the team at LNE is the so-called R-pump.[21] In this device, the impedance fabricated on the same chip as the SET pump is used to suppress the undesirable cotunneling processes that degrade the accuracy of the device. This device concept has been proposed in Ref. 22 as an application of the results of the analysis of the effects of the electromagnetic fluctuations on the inelastic cotunneling. Excellent agreement between the theory and the experimental results was reported by Zorin *et al.*[23] Further analysis of the questions raised in that paper with regard to the particular transmission line used in the experiment was analyzed in Ref. 24. Based on this experiment, the R-pump device was fabricated at PTB.[25] As in the conventional pump, the electron transfer through the circuit implementing the R-pump is achieved by the periodic modulation of the islands' potentials. Under the most favorable conditions of low temperature and suitable fabrication to achieve low offset charge fluctuations, the main processes that limit the accuracy of the device are the cotunneling processes. In a three-junction R-pump, as the islands' potentials change, instead of an electron being transferred through one junction in the forward direction, charge can be transferred in the backward direction through the other two junctions. Analogous to macroscopic tunneling of the phase difference in Josephson junctions, these processes were termed macroscopic tunneling of charge (q-MQT) by Averin and Nazarov.[26] Two types of such processes exist. Either two electrons tunnel simultaneously through two junctions or one electron tunnels through both junctions. In the first case, an electron-hole excitation remains on the island, so those processes are called inelastic cotunneling. Processes involving one electron are called elastic cotunneling. Both types of processes were considered in Ref. 26 in the absence of environmental fluctuations. Potential modulation can be achieved by harmonic or triangular drives, each having some advantage. For harmonic gates, theoretical analysis of the accuracy of an R-pump was performed in Ref. 27. For triangular drives, it is important to include the rates of the elastic cotunneling processes, which dominate over inelastic cotunneling at low voltages and temperatures $eV, k_BT \ll \sqrt{\Delta E_c}$,[26] where Δ is the mean level spacing of the island and E_c is the charging energy of the island. Deriving expressions for elastic cotunneling rates that can be used for this purpose is the subject of this paper. We consider a dot in a metallic regime (the transport mean-free path or the size of the dot is much larger than the Fermi wavelength) since the R-pump used in the experiment is typically fabricated from aluminum by the standard double-angle evaporation technique.

Inelastic cotunneling has also been analyzed by Golubev and Zaikin.[28] Starting from a path-integral expression for the partition function and imaginary-time action for a series of junctions, they obtained a result for the inelastic cotunneling rate from the imaginary part of the free energy, generalizing our result for the double junction[22] to the case of an array of N junctions in series. At the Coulomb blockade threshold voltage, the expression for inelastic cotunneling obtained in the lowest order of perturbation theory in the tunnel Hamiltonian diverges. A diagrammatic technique has been developed in Ref. 29 that removes this divergence by summing perturbative terms to infinite order. Cotunneling is also important for all SET devices that are being investigated for future information processing and storage applications.[30] For example, the effect of an on-chip resistor on inelastic cotunneling, and therefore on the retention time of the memory cell, was investigated by Lotkhov *et al.*[31] Agreement with the theory of inelastic cotunneling, in the case of a low impedance environment, was reported in Ref. 32 where the results of measurements of time-resolved single-electron tunneling events in a single-electron trap are presented. Elastic cotunneling has also been studied as a way to probe entanglement[33] in relation to quantum information processing applications.

The remainder of the paper is organized as follows. In Sec. II we introduce the method; we consider a double junction system in the Coulomb blockade regime with a general impedance environment; and we derive an expression for the elastic cotunneling rate. In Sec. III we apply the general result to the specific geometry of the rectangular island and obtain analytic results for low- and high-environment impedance. Section IV is devoted to summary.

II. METHOD

We use the standard model for electron transport through a double junction system. The Hamiltonian that includes the electromagnetic environment degrees of freedom reads

$$H = H_0 + H_T, \tag{3}$$

where $H_0 = \Sigma_{i=S,I,D} H_i + H_{\text{env}}$ describes the decoupled electrodes (source, drain, and the island) as well as the environment. The quasiparticle energies in the outer electrodes include fluctuations of the voltage sources due to circuit impedance $Z(\omega)$, $H_i = \Sigma_l (\epsilon_l + eV_i) c_{l,i}^{\dagger} c_{l,i}$ $(i=S,D)$. The electron-electron interaction on the island is approximated by a charging energy term $H_I = \Sigma_\alpha \epsilon_\alpha c_\alpha^{\dagger} c_\alpha + Q^2/2C_\Sigma$, where Q is the excess island charge and $C_\Sigma = C_1 + C_2$ is the island capacitance. The linear circuit elements are described by the Bose operators corresponding to the normal modes of the environment $H_{\text{env}} = \Sigma_\alpha \hbar \omega_\alpha b_\alpha^{\dagger} b_\alpha$.[34,35] We are considering the tunneling perturbatively; that is, we consider high-resistance junctions, $R_i \gg \max[R_K, \text{Re}\, Z(\omega)]$, for frequencies $\omega \le eV_i/\hbar$. It is convenient to proceed in the interaction picture with respect to H_0, where the tunneling Hamiltonian is given by

$$H_T = H_{T1} + H_{T2}, \quad H_{Ti} = H_i^{+} + H_i^{-},$$

$$H_1^+ = \sum_{\alpha,p} T_{\alpha p} c_\alpha^\dagger(t) c_p(t) e^{-i\varphi_1(t)}, \quad H_i^- = (H_i^+)^\dagger. \tag{4}$$

The phase $\varphi_i = \int dt e\widetilde{V}_i(t)/\hbar = \varphi_i^{(\mathrm{cl})}(t) + \widetilde{\varphi}_i(t)$ consists of the classical, $\varphi_i^{(\mathrm{cl})} = eV_i t/\hbar$, and the quantum fluctuation parts, $\widetilde{\varphi}_i$, the latter parts being treated as operators. The voltage across each junction is given by

$$V_i = \frac{V}{2} + (-1)^i \frac{Q_0}{C_\Sigma}, \tag{5}$$

where $Q_0 = C_G V_G - e[C_G V_G/e]$ is the offset charge provided by the gate source.

The expectation value of the current is given by

$$I = \langle S^{-1}(t,-\infty)\hat{I}(t)S(t,-\infty)\rangle, \tag{6}$$

where $S = \mathcal{T}\exp[(-i/\hbar)\int_{-\infty}^t d\tau H_T(\tau)]$ and the current operator is given by $\hat{I}(t) = (ie/\hbar)[H_1^-(t) - H_1^+(t)]$. The averaging is taken over the equilibrium electron system as well as the environment.

Since only even powers of the tunneling Hamiltonian contribute in Eq. (6), and we are considering the Coulomb blockade regime $[V < V_t = (e/2 - |Q_0|)/\max(C_1, C_2)]$, the lowest nonvanishing contribution to the current is given by

$$I = \frac{1}{(i\hbar)^3}\Bigg\{\int_{-\infty}^t d\tau \int_{-\infty}^t d\tau' \int_{-\infty}^{\tau'} d\tau'' [\langle H_T(\tau'')H_T(\tau')\hat{I}(t)H_T(\tau)\rangle - \langle H_T(\tau)\hat{I}(t)H_T(\tau')H_T(\tau'')\rangle] + \int_{-\infty}^t d\tau \int_{-\infty}^{\tau} d\tau' \int_{-\infty}^{\tau'} d\tau'' [\langle \hat{I}(t)H_T(\tau)H_T(\tau')H_T(\tau'')\rangle - \langle H_T(\tau'')H_T(\tau')H_T(\tau)\hat{I}(t)\rangle]\Bigg\}. \tag{7}$$

Collecting the terms that correspond to q-MQT, we have

$$I = \frac{2e}{\hbar^4}\sum_{i,j=1,2(i\neq j)} \mathrm{Re}\Bigg\{\int_{-\infty}^t d\tau \int_{-\infty}^t d\tau' \times \int_{-\infty}^{\tau'} d\tau'' [\langle H_i^-(\tau'')H_j^-(\tau')H_1^+(t)H_2^+(\tau)\rangle - \langle H_2^+(\tau)H_1^+(t)H_i^-(\tau')H_j^-(\tau'')\rangle] + \int_{-\infty}^t d\tau \int_{-\infty}^{\tau} d\tau' \int_{-\infty}^{\tau'} d\tau'' [\langle H_1^+(t)H_2^+(\tau)H_i^-(\tau')H_j^-(\tau'')\rangle - \langle H_i^-(\tau'')H_j^-(\tau')H_2^+(\tau)H_1^+(t)\rangle]\Bigg\}. \tag{8}$$

The phase operators $\widetilde{\varphi}_i$ are represented by a linear combination of Bose operators corresponding to the environmental modes and as such they commute with the quasiparticle operators. Using Wick's theorem and the orthogonality of unperturbed states, we get for the first term in the integrand in Eq. (8)

$$\langle H_1^-(\tau'')H_2^-(\tau')H_1^+(t)H_2^+(\tau)\rangle = \sum_{p,\alpha,\beta,m} T_{\alpha p}^* T_{m\beta} f(\epsilon_p)[1-f(\epsilon_\alpha)]f(\epsilon_\beta)[1-f(\epsilon_m)] \times e^{i\epsilon_p(\tau''-t)/\hbar} e^{-i\epsilon_m(\tau'-\tau)/\hbar} (T_{\alpha p} T_{m\beta}^* e^{-i\epsilon_\alpha(\tau''-t)/\hbar} e^{i\epsilon_\beta(\tau'-\tau)/\hbar} - T_{\beta p} T_{m\alpha}^* e^{-i\epsilon_\alpha(\tau''-\tau')/\hbar} e^{i\epsilon_\beta(t-\tau)/\hbar}) \times \langle e^{i\varphi_1(\tau'')} e^{i\varphi_2(\tau')} e^{-i\varphi_1(t)} e^{-i\varphi_2(\tau)}\rangle, \tag{9}$$

where $f(\epsilon)$ is the Fermi distribution function and the trace is over the environmental states.

The terms in Eq. (8) that are of a similar form to the first term on the right-hand side of Eq. (9) are proportional to the absolute values of the tunneling amplitudes and describe the simultaneous tunneling of two electrons through both junctions. The second term in Eq. (9) depends on the phases of the tunnel matrix elements and describes the elastic cotunneling process whereby one electron tunnels through both junctions. By collecting elastic cotunneling terms, we can write

$$I = e(\gamma^+ - \gamma^-), \tag{10}$$

where the forward tunneling rate is given by

$$\gamma^+ = \frac{2}{\hbar^4}\mathrm{Re}\Bigg\{\sum_{p,\alpha,\beta,m} T_{p\alpha}T_{\alpha m}T_{m\beta}T_{\beta p} f(\epsilon_p)[1-f(\epsilon_m)] \int_{-\infty}^t d\tau \int_{-\infty}^t d\tau' \times \int_{-\infty}^{\tau'} d\tau'' \langle (-[1-f(\epsilon_\alpha)] e^{i\epsilon_p(\tau''-t)/\hbar} e^{-i\epsilon_\alpha(\tau''-\tau')/\hbar} e^{i\epsilon_\beta(t-\tau)/\hbar} e^{-i\epsilon_m(\tau'-\tau)/\hbar} e^{i\varphi_1(\tau'')} e^{i\varphi_2(\tau')} + f(\epsilon_\alpha) e^{i\epsilon_p(\tau'-t)/\hbar} e^{-i\epsilon_\alpha(\tau'-\tau'')/\hbar} e^{i\epsilon_\beta(t-\tau)/\hbar} e^{-i\epsilon_m(\tau''-\tau)/\hbar} e^{i\varphi_2(\tau'')} e^{i\varphi_1(\tau')}) (e^{-i\varphi_1(t)} e^{-i\varphi_2(\tau)} f(\epsilon_\beta) + \Theta(\tau-\tau') e^{-i\varphi_2(\tau)} e^{-i\varphi_1(t)}[1-f(\epsilon_\beta)])\rangle\Bigg\}. \tag{11}$$

For $Z(\omega)\to 0$, only the classical part of the phase $\varphi_i^{(\mathrm{cl})}(t)$ enters the exponents, and the above formula reduces to the case of Averin and Nazarov.[26] Using the Baker-Hausdorff formula and the properties of the harmonic system, the averages in the above formula can be expressed in terms of the phase-phase correlation functions

$$J_{i,j}(t) = \langle(\tilde{\varphi}_i(t) - \tilde{\varphi}_i(0))\tilde{\varphi}_j(0)\rangle = \frac{1}{R_k}\int_{-\infty}^{\infty} d\omega \frac{\mathrm{Re}\, Z_{i,j}(\omega)}{\omega}\left\{\coth\left(\frac{\hbar\omega}{2k_BT}\right)[\cos(\omega t) - 1] - i\sin(\omega t)\right\}, \tag{12}$$

where the second equality is written with the aid of the fluctuation-dissipation theorem,[36] and where we have introduced an impedance $Z_{i,j}$ that determines the linear response of the circuit $V_i(\omega)=Z_{i,j}(\omega)I_j(\omega)$ in the absence of tunneling. Since elastic cotunneling is dominant at low temperatures, $k_BT \ll e^2/C_\Sigma$, while at higher temperatures inelastic cotunneling dominates, we consider the zero-temperature case below. Separating the pole of the impedance we can write

$$J_{i,i}(t) = -i\omega_c t + \bar{J}_i(t), \tag{13}$$

where $\omega_c = e^2/2C_\Sigma\hbar$, and $\bar{J}_i(t)$ is the correlation function corresponding to an equivalent circuit of a single junction with capacitance $C_{sj}=C_\Sigma C_i/C_j$ embedded in a circuit with external impedance $Z_{sj}=Z(C_j/C_\Sigma)^2$.

Similarly, for $i\neq j$, we can write

$$J_{i,j}(t) = i\omega_c t + \bar{K}(t), \tag{14}$$

where $\bar{K}_i(t)$ is the correlation function corresponding to an equivalent circuit of a single junction with capacitance $C'_{sj}=C_\Sigma$ embedded in a circuit with external impedance $Z'_{sj}=ZC_1C_2/C_\Sigma^2$.

In order to simplify the calculations of $\bar{J}_i(t)$ and $\bar{K}(t)$, in integrals of the form as those in Eq. (12) we approximate the real part of the corresponding impedance by $\mathrm{Re}\, Z_{sj}(0)\exp(-\omega/\Omega_0)$, where $\Omega_0 = (\mathrm{Re}\, Z_{sj}(0)C_{sj})^{-1}$, which is legitimate at low frequencies, $\omega \ll \Omega_0$. Since we are interested in the effects of dissipation on cotunneling, we also assume that the circuit impedance is Ohmic at low frequencies, $Z(\omega\ll\Omega_0)\sim \mathrm{Re}\, Z(0)=R$, in accordance with a typical experimental situation.[23] By performing analytic continuation to imaginary times in integrals over τ and τ'' in Eq. (11) we obtain

$$\gamma^+ = \frac{2\pi}{\hbar}\frac{1}{(\hbar\Omega_0)^3}\frac{1}{\Gamma(2z)}\sum_{p,\alpha,\beta,m} T_{p,\alpha}T_{\alpha,m}T_{m,\beta}T_{\beta p}f(\epsilon_p) \times[1-f(\epsilon_m)]F(\epsilon_\alpha,\epsilon_p,\epsilon_m)F(\epsilon_\beta,\epsilon_p,\epsilon_m) \times\left(\frac{eV+\epsilon_p-\epsilon_m}{\hbar\Omega_0}\right)^{2z-1} e^{-\frac{eV+\epsilon_p-\epsilon_m}{\hbar\Omega_0}}\Theta(eV+\epsilon_p-\epsilon_m),$$

$$F(\epsilon,\epsilon_p,\epsilon_m) = [1-f(\epsilon)]U\left(1,2+\frac{2C_1C_2z}{(C_1+C_2)^2},\frac{E_1-\epsilon_p+\epsilon}{\hbar\Omega_0}\right) - f(\epsilon)U\left(1,2+\frac{2C_1C_2z}{(C_1+C_2)^2},\frac{E_2+\epsilon_m-\epsilon}{\hbar\Omega_0}\right), \tag{15}$$

where $z=\mathrm{Re}\, Z(0)/R_K$ and $U(a,b,c)$ is Kummer's confluent hypergeometric function of the second kind.[37] Energies $E_i = \hbar\omega_c - eV_i$ correspond to the Coulomb energy increase in the system from the initial state to the intermediate state in the cotunneling process by electron tunneling through junction i. The expression for the backward tunneling rate, γ^-, is obtained from Eq. (15) by replacing $V\to -V$, $E_i\to E_i+eV$, $\epsilon_p \leftrightarrow \epsilon_m$. In order to describe virtual electron propagation on the island, it is convenient to express the tunneling amplitudes in the coordinate representation

$$T_{\alpha p} = \int d\mathbf{y}\int d\mathbf{x}T(\mathbf{y},\mathbf{x})\psi^*_\alpha(\mathbf{y})\psi_p(\mathbf{x}). \tag{16}$$

In that way we can write

$$\gamma^+ = \frac{2\pi}{\hbar}\frac{1}{(\hbar\Omega_0)^3}\frac{1}{\Gamma(2z)}\int d\epsilon_p d\epsilon_\alpha d\epsilon_\beta d\epsilon_m Q(\epsilon_p,\epsilon_\alpha,\epsilon_\beta,\epsilon_m) \times f(\epsilon_p)F(\epsilon_\alpha)F(\epsilon_\beta)[1-f(\epsilon_m)] \times\left(\frac{eV+\epsilon_p-\epsilon_m}{\hbar\Omega_0}\right)^{2z-1} e^{-\frac{eV+\epsilon_p-\epsilon_m}{\hbar\Omega_0}}\Theta(eV+\epsilon_p-\epsilon_m), \tag{17}$$

where $F(\epsilon)\equiv F(\epsilon,0,0)$ and the function $Q(\epsilon_p,\epsilon_\alpha,\epsilon_\beta,\epsilon_m)$ is given in terms of Green's functions

$$Q(\epsilon_p,\epsilon_\alpha,\epsilon_\beta,\epsilon_m) = \frac{1}{\pi^4}\int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{y}_1 d\mathbf{y}_2 d\mathbf{y}_3 d\mathbf{y}_4 d\mathbf{z}_1 d\mathbf{z}_2 \times T(\mathbf{y}_1,\mathbf{x}_1)T^*(\mathbf{y}_2,\mathbf{x}_2)T(\mathbf{y}_3,\mathbf{z}_1)T^*(\mathbf{y}_4,\mathbf{z}_2) \times \mathrm{Im}\, G^R(\mathbf{x}_2,\mathbf{x}_1;\epsilon_p)\mathrm{Im}\, G^R(\mathbf{y}_4,\mathbf{y}_2;\epsilon_\alpha) \times\mathrm{Im}\, G^R(\mathbf{y}_1,\mathbf{y}_3;\epsilon_\beta)\mathrm{Im}\, G^R(\mathbf{z}_1,\mathbf{z}_2;\epsilon_m), \tag{18}$$

where the points $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are located on the left electrode, island, and right electrode, respectively. By writing $\mathrm{Im}\, G^R = (G^R - G^A)/2i$, function Q can be expressed in terms of the averaged products $\langle G^A G^R\rangle$ (Diffuson and Cooperon ladders). In such a way $Q(\epsilon_p,\epsilon_\alpha,\epsilon_\beta,\epsilon_m)$ can be represented in terms of the quasiclassical probability $P(\mathbf{y}_1,\mathbf{n}_1,0;\mathbf{y}_2,\mathbf{n}_2,t)$ to find an electron at time t at point $\mathbf{y}_2$ with momentum $\mathbf{p}_2 = p_F\mathbf{n}_2$ given that at the initial moment it was in the state $(\mathbf{y}_1,\mathbf{n}_1)$, assuming an isotropic Fermi surface.[38] The magnetic-field dependence of mesoscopic fluctuations of elastic cotunneling has been considered by Aleiner and Glazman.[39] Here, we are not considering the effects of the magnetic field and spin-orbit scattering, and therefore in our case Diffuson and Cooperon correlators are equal. In the weak disorder limit (that is, when the electron wavelength is small compared to the elastic mean-free path, $\lambda \ll l_e$, and for sufficiently large length scales, $l_e \ll L_{\min}$) the probability P does not depend on $\mathbf{n}$, and it satisfies the diffusion equation. We obtain from Eqs. (17) and (18)

$$\gamma^{+}=\frac{1}{(2\pi)^{2}\Gamma(2+2z)e^{4}\nu\hbar\Omega_{0}}\left(\frac{eV}{\hbar\Omega_{0}}\right)^{1+2z}\int d\epsilon_{\alpha}d\epsilon_{\beta}F(\epsilon_{\alpha})F(\epsilon_{\beta})\times\int d\mathbf{x}_{1}d\mathbf{x}_{2}g_{1}(\mathbf{x}_{1})g_{2}(\mathbf{x}_{2})\int dte^{i(\epsilon_{\alpha}-\epsilon_{\beta})t/\hbar}P(\mathbf{x}_{1},0;\mathbf{x}_{2},|t|), \tag{19}$$

for $eV\ll\hbar\Omega_0$. In the above formula $g_i(\mathbf{x}_i)$ is the conductance per unit area of junction i, and ν is the density of states.

III. RECTANGULAR ISLAND

The probability of quantum diffusion depends on the shape of the island. As a specific example, we consider a rectangular box of volume $L_xL_yL_z$ with junctions separated by distance L_y and of material characterized by the diffusion constant D. We also assume that $g_i(\mathbf{x}_i)$ are constant $[g_i(\mathbf{x}_i)=g_i]$. In order to obtain analytic expressions, we concentrate on the limits of low, $z\ll 1$, and high circuit impedance, $z\gg 1$. By solving the diffusion equation subject to the Neumann boundary conditions (during the diffusion, the electron does not leave the island), we get

$$P(x_1,z_1,0;x,z,t)=\frac{1}{L_xL_yL_z}\sum_{l,m,n=0}^{\infty}(2-\delta_{l,0})(2-\delta_{m,0})(2-\delta_{n,0})\times(-1)^{m}\exp[-(l^2/\tau_x+m^2/\tau_y+n^2/\tau_z)\pi^2t]\times\cos(l\pi x_1/L_x)\cos(n\pi z_1/L_z)\times\cos(l\pi x/L_x)\cos(n\pi z/L_z). \tag{20}$$

We distinguish two regimes with respect to the characteristic timescale $1/\omega_c$. For sufficiently short times, an electron diffuses as in an infinite medium, while for longer times the probability of reaching the boundary is not negligible.

We first consider the case of low circuit impedance, $z\ll 1$. By using $U(a,b,c)\sim\Gamma(b-1)/\Gamma(a)c^{b-1}$, and by first integrating over time and over two island surfaces perpendicular to the transport direction, we obtain for $\omega_c\tau_{\max}\ll 1$ (ergodic regime), where $\tau_{\max}=[\max(L_x,L_y,L_z)]^2/D$,

$$I=\frac{\hbar G_1G_2\Delta}{4\pi e^3}\left(\frac{eV}{\hbar\Omega_0}\right)^{1+2z}\left[\left(\frac{\hbar\Omega_0}{E_1}\right)^{1+[4C_1C_2z/(C_1+C_2)^2]}+\left(\frac{\hbar\Omega_0}{E_2}\right)^{1+[4C_1C_2z/(C_1+C_2)^2]}\right], \tag{21}$$

where $G_i=\int d\mathbf{x}_ig_i(\mathbf{x}_i)$ is the total conductance of the junction i, and $\Delta=2/\nu L_xL_yL_z$ is the level spacing. In the opposite limit of short tunneling times (free diffusion regime), $\omega_c\tau_{\min}\gg 1$, we obtain

$$I=\frac{2\hbar G_1G_2\Delta}{\pi^2e^3\Omega_0\tau_y}\left(\frac{eV}{\hbar\Omega_0}\right)^{1+2z}\left[\left(\frac{\hbar\Omega_0}{E_1}\right)^{2\{1+[2C_1C_2z/(C_1+C_2)^2]\}}+2\left(\frac{(\hbar\Omega_0)^2}{E_1E_2}\right)^{1+[2C_1C_2z/(C_1+C_2)^2]}+\left(\frac{\hbar\Omega_0}{E_2}\right)^{2\{1+[2C_1C_2z/(C_1+C_2)^2]\}}\right]. \tag{22}$$

Sometimes it might be more convenient to reverse the order of integration in Eq. (19). We use this approach in the case of high impedance, $z\gg 1$. Using the large argument behavior $U(a,b,c)\sim c^{-1}$, and first integrating over energies we get

$$\gamma^{+}=\frac{\hbar G_1G_2\Delta}{\Gamma(2+2z)e^4\Omega_0}\left(\frac{eV}{\hbar\Omega_0}\right)^{1+2z}\int_0^{\infty}dt|F(t)|^2\vartheta_4(0,t/\tau_y), \tag{23}$$

where $\vartheta_4(u,x)$ is the elliptic theta function, and $F(t)$ is given by

$$F(t)=(2\pi\hbar)^{-1}\int_{-\infty}^{\infty}d\epsilon e^{i\epsilon t/\hbar}F(\epsilon)=\frac{\Omega_0}{2\pi}\{g(E_1t/\hbar)-g(E_2t/\hbar)+i[f(E_1t/\hbar)+f(E_2t/\hbar)]\},$$

$$g(\epsilon)=-[\sin(\epsilon)si(\epsilon)+\cos(\epsilon)ci(\epsilon)],$$

$$f(\epsilon)=\sin(\epsilon)ci(\epsilon)-\cos(\epsilon)si(\epsilon), \tag{24}$$

where $si(\epsilon)$ and $ci(\epsilon)$ are the sine and cosine integral functions. For $\omega_c\tau_{\max}\ll 1$ we get

$$I=\frac{\hbar G_1G_2\Delta}{4\pi e^2\Gamma(2+2z)}\left(\frac{eV}{\hbar\Omega_0}\right)^{2z}\left(\frac{1}{E_1}+\frac{1}{E_2}\right)V. \tag{25}$$

In the opposite limit, $\omega_c\tau_{\min}\gg 1$, using the asymptotic behavior for large arguments of auxiliary functions in Eq. (24), $g(\epsilon)\sim\epsilon^{-2}, f(\epsilon)\sim\epsilon^{-1}$, we get

$$I=\frac{2\hbar^2G_1G_2\Delta}{\Gamma(2+2z)\pi^2e^2\tau_y}\left(\frac{eV}{\hbar\Omega_0}\right)^{2z}\left(\frac{1}{E_1}+\frac{1}{E_2}\right)^2V. \tag{26}$$

For cases when the characteristic diffusion time is of the order of $1/\omega_c$, cotunneling current can be obtained numerically from Eq. (23). Since we have used the limiting behavior of the diffusion probability for short and long times, the results above will be applicable for an arbitrary shape of the island as long as the junction conductances are constant across the junction surfaces. When this is not the case, the general expression (19) and, in particular, the power-law suppression are still applicable.

IV. SUMMARY

We have investigated the influence of circuit impedance on elastic cotunneling at zero temperature. The derived expressions are useful for accuracy considerations of the

R-pump used in the quantum metrological triangle experiment. When triangular gates are used for such a pump, the potentials of the islands are swept over the whole Coulomb blockade region, and therefore both elastic and inelastic cotunneling processes have to be taken into account. By comparing the results (21), (22), (25), and (26) with the results for inelastic cotunneling [that is, formulas (25) and (26) in Ref. 22], we obtain the crossover voltage where inelastic cotunneling starts to dominate over elastic cotunneling, $eV_{\rm cr} \simeq \sqrt{\Delta \min(\hbar\omega_c, \hbar/\tau_y)}$, independent of the circuit impedance, which therefore coincides with the result of Averin and Nazarov.[26] The obtained non-Ohmic power-law behavior of the tunneling current can be interpreted as renormalization of the intermediate state energies in the cotunneling process due to the quantum fluctuations of the environment. The derived expressions are also of interest for the accuracy considerations of classical and quantum information processing devices that employ the SET effect.

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